What is a Category? Sheaf? Topos? Semantics?

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Proof and Computation
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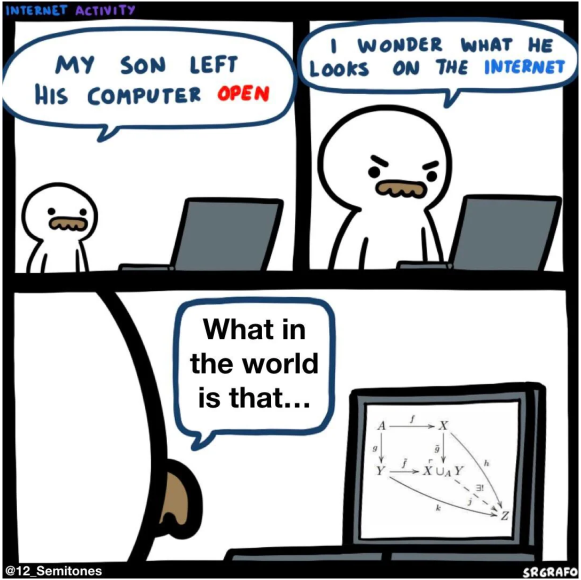
What we hope to achieve

- Professor Coquand's talk had a high barrier to entry
- Explain some motivation / intuition for Categories / Sheaves / Toposes
- Introduce the definitions
- Say what these things are used for
- My contribution: Categories & Sheaves
- Apologies: these slides were rushed

What is a Category?

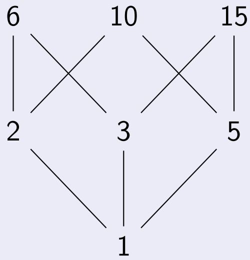


Preorders

Definition

A **pre-order / quasi-order (poset)** is a pair $(P, \leq)$ where:

- P is a set, and
- $\leq$ is a binary relation on P that is:
  - Reflexive: $\forall x \in P, x \leq x$,
  - Transitive: $\forall x, y, z \in P, (x \leq y \text{ and } y \leq z) \implies x \leq z$.



Proof relevant Posets

A Category is a Poset with evidence – Alex Kavvos

- A proposition $x \leq y$ should have a proof $p : x \leq y$
- It might have more that one
- Reflexivity becomes that for any x, there is a proof of it

$$x \mapsto \mathrm{id}_x : x \leq x$$

- Transitivity says there is a way to combine evidence

$$p : x \leq y, q : y \leq z \mapsto q \circ p : x \leq z$$

- These constructions should interact sensibly

Categories

Definition (Category)

A *category* $\mathcal{C}$ consists of a collection of *objects* $A, B, C, \ldots$ and a collection of *arrows* $f, g, h, \ldots$ such that

- Any arrow $f : \text{dom}(f) \to \text{cod}(f)$ has two objects $\text{dom}(f)$ and $\text{cod}(f)$.
- Given two compatible arrows $f : B \to C$, $g : A \to B$, there is a *composite arrow* $f \circ g : A \to C$.
- There is an *identity arrow* for every object A, $\text{id}_A : A \to A$.
- $\circ$ is associative: $f \circ (g \circ h) = (f \circ g) \circ h$,
- and the id_A are identities for composition: $f \circ \text{id} = f = \text{id} \circ f$

Categories (Examples)

Example

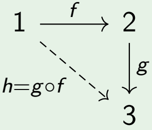
Most mathematical objects can be bundled into categories: **Set**, **Rel**, $\mathbf{Vect}_{\mathbb{K}}$, ...

Example

Various objects can be thought of as categories: monoids, posets, ...

Example (3)

You can freely generate categories from graphs



Monotone functions

Once you have an object, you should ask "what are the morphisms between them?"

Definition (Monotone Functions)

A function $f : (P, \leq_P) \to (Q, \leq_Q)$ between two posets $(P, \leq_P)$ and $(Q, \leq_Q)$ is called *monotone* (or *order-preserving*) if:

$$\forall x, y \in P,\ x \leq_P y \implies f(x) \leq_Q f(y).$$

What happens when we add evidence?

Monotone Functions with Evidence

- We still send objects to objects $x \mapsto f(x)$
- but need to send proofs $p : x \leq y$ to proofs $f(p) : f(x) \leq f(y)$
- and we should do something sensible with id and $\circ$
- we have two proofs $id_{f(x)}, f(id_x) : f(x) \leq f(x)$. Make them equal!
- and if we have proofs $p : x \leq y$, $g : y \leq z$, there are two proofs $f(q \circ p), f(q) \circ f(p) : f(x) \leq f(z)$. Make them equal!

Functors

Definition (Functor)

A *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of mappings on objects and arrows such that

- Given any $\mathcal{C}$-object X, $F(X)$ is a $\mathcal{D}$-object.
- Given any $\mathcal{C}$-arrow $f : A \to B$, $F(f) : F(A) \to F(B)$ is a $\mathcal{D}$-arrow.
- $F(f \circ g) = F(f) \circ F(g)$
- $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$

Functor Examples

- Identity Functor
- Forgetful Functors
- Forgetful Functors
- Polynomial Functors
- Hom Functors
- Group Actions
- Automata (Quentin's Lightning Talk)
- Most things topologists care about
  - Fundamental Group
  - Homology
  - One Point Compactification
  - ...

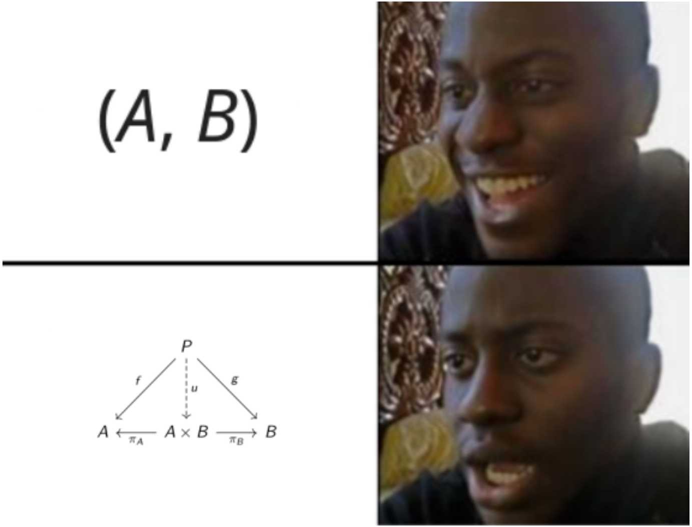
Thinking Categorically

Arrows are more important than Objects – Every Category Theorist

Problem: How to define useful structures only by looking at arrows?

Case Study: Products

Categorical Definitions: Products



Categorical Definitions: Products

- When we have a pair (x, y) we want to be able to get at the x and y.
- In **Set**, we have projection functions $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$
- We can compose these with functions that output pairs $u : P \to X \times Y$

$$\pi_X \circ u : P \to X$$

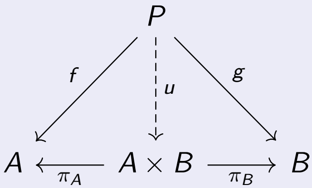
$$\pi_Y \circ u : P \to Y$$

- Functions into pairs are equivalent to pairs of functions
- Can this help?

Categorical Definitions: Products

Definition (Product)

Let $\mathcal{C}$ be a category and let A, B be objects of $\mathcal{C}$. A *product* of A and B consists of an object $A \times B$ and arrows $\pi_A : A \times B \to A$, $\pi_B : A \times B \to B$ which is *universal*.
This means that for any other triple $(P, f : P \to A, g : P \to B)$ there exists a *unique* map making the following diagram commute.



Products are unique up to unique isomorphism.

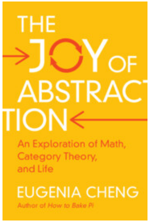
Categorical Definitions: Products

Not all pairs of objects in a category have products and those that do don't necessarily "look like" pairs

- Set – the usual Cartesian product
- Group – direct product of groups
- Top – the product space
- Graphs – tensor product of graphs
- Rel – disjoint union of relations
- A poset – the meet of two objects
- ...

Some Resources

- 27 Unhelpful Facts About Category Theory – Oliver Lugg
- Category Theory in Context – Emily Riehl
- Basic Category Theory – Tom Leinster
- Seven Sketches in Compositionality – Brendan Fong, David Spivak
- The Joy of Abstraction – Eugenia Cheng (Not Free)



Break for Questions

What is a Sheaf?

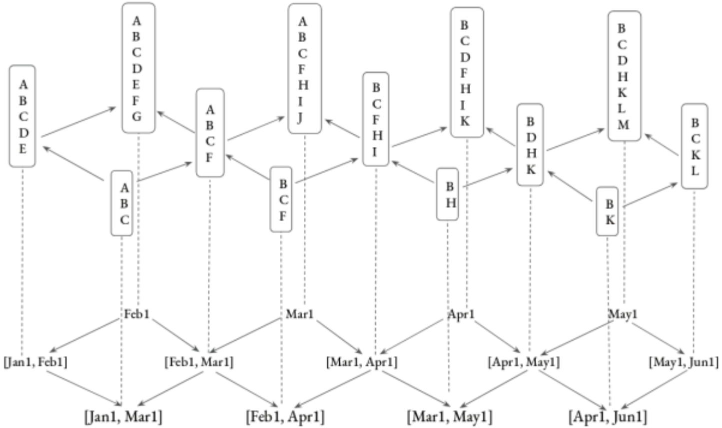


Figure: Example 56 – Sheaf Theory Through Examples

Scaring Undergrads in Analysis (Manifolds)

Definition 2.5.1. *Let $k \leq n \in \mathbb{N}$. A* k-dimensional smooth manifold *(abbreviated as* smooth k-manifold*) in $\mathbb{R}^n$ is a subset M of $\mathbb{R}^n$ along with a family $(\Phi_i)_{i \in I}$ of smooth embeddings*

$$\Phi_i : U_i \to M$$

called charts, *such that $U_i \subset \mathbb{R}^k$ is an open set for each i, $\Phi_i(U_i)$ is relatively open in M (i.e., it is the intersection of an open set in $\mathbb{R}^n$ and M), and*

$$M = \bigcup_{i \in I} \Phi_i(U_i).$$

It is oriented *if for each $i, j \in I$, if $V_{i,j} := \Phi_i(U_i) \cap \Phi_j(U_j)$ is non-empty, then the maps*

$$\Phi_i|_{\Phi_i^{-1}(V_{i,j})} \quad \text{and} \quad \Phi_j|_{\Phi_j^{-1}(V_{i,j})}$$

have the same orientation. The manifold is compact *if M is a compact subset of $\mathbb{R}^n$.*

Figure: My Analysis IV Lecture Notes – Aaron Tikuisis

Scaring Undergrads in Analysis (Analytic Continuation)

Definition (Analytic Continuation)

Suppose $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is an analytic functions. f can be *analytically continued* to a region D' which intersects D if there exists an analytic function $g : D' \to \mathbb{C}$ such that $f = g$ on $D \cap D'$

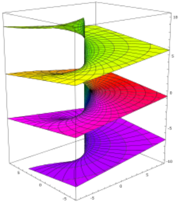


Figure: Riemann Surface of Natural Logarithm – Wikipedia

Global Sections

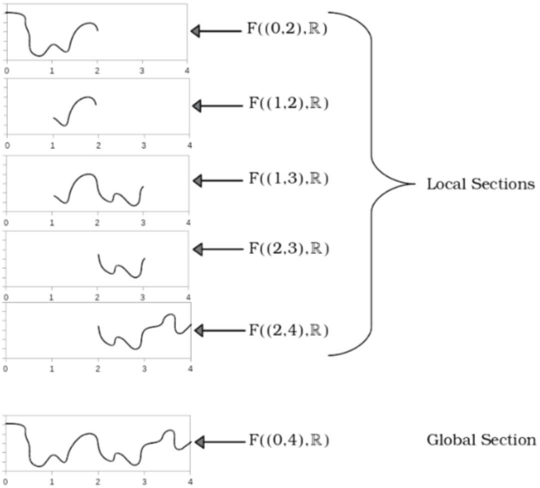


Figure: Example 125 – Sheaf Theory Through Examples

The Definition™

Definition (Sheaf)

A *sheaf* on a topological space X is a presheaf (contravariant functor)

$$\mathcal{F} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Sets}$$

satisfying the following properties:
1. **Locality:** Suppose U is an open set, $\{U_i\}_{i \in I}$ is an open cover of U with $U_i \subseteq U$ for all $i \in I$, and $s, t \in \mathcal{F}(U)$ are sections. If $s\big|_{U_i} = t\big|_{U_i}$
2. **Gluing:** Suppose U is an open set, $\{U_i\}_{i \in I}$ is an open cover of of U with $U_i \subseteq U$ for all $i \in I$, and $\{s_i \in \mathcal{F}(U_i)\}_{i \in I}$ is a family of sections. If $s_i\big|_{U_i \cap U_j} = t_i\big|_{U_i \cap U_j}$ for all $i, j \in I$, then there exist a section $s \in \mathcal{F}(U)$ such that $s\big|_{U_i} = s_i$ for all $i \in I$.

Topological Spaces

Definition (Sheaf)

A *sheaf* on a topological space X is a ...

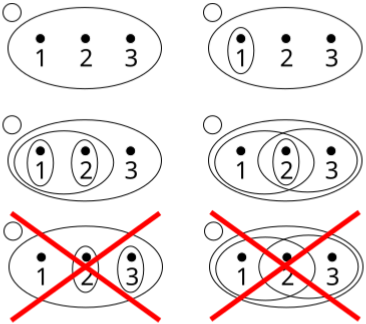


Figure: From Wikipedia

Unravelling the definition

Definition (Sheaf)

A *sheaf* on a topological space X is a presheaf (contravariant functor)

$$\mathcal{F} : \mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{Sets}$$

satisfying Locality and Gluing.

- $\mathcal{F}$ maps each open set U to a set $\mathcal{F}(U)$.
- If $U \subseteq V$, then we have a function $\rho_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$.
- $\rho_U^U = \mathrm{id}_U$.
- If $U \subseteq V \subseteq W$, then we have a function $\rho_U^W = \rho_V^W \circ \rho_U^V$.
- Sometimes we write $s\big|_U := \rho_U^V(s)$

Unravelling the definition

Definition (Sheaf)

A *sheaf* on a topological space X is a presheaf $\mathcal{F} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Sets}$ satisfying
Locality: Suppose U is an open set, $\{U_i\}_{i \in I}$ is an open cover of U with $U_i \subseteq U$ for all $i \in I$, and $s, t \in \mathcal{F}(U)$ are sections. If $s\big|_{U_i} = t\big|_{U_i}$ for all $i \in I$, then $s = t$.
and Gluing.

To see if two objects agree everywhere (globally), check they agree on enough neighbourhoods (locally).

Unravelling the definition

Definition (Sheaf)

A *sheaf* on a topological space X is a presheaf $\mathcal{F} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Sets}$ satisfying locality and **Gluing:** Suppose U is an open set, $\{U_i\}_{i \in I}$ is an open cover of of U with $U_i \subseteq U$ for all $i \in I$, and $\{s_i \in \mathcal{F}(U_i)\}_{i \in I}$ is a family of sections. If $s_i\big|_{U_i \cap U_j} = t_i\big|_{U_i \cap U_j}$ for all $i, j \in I$, then there exist a section $s \in \mathcal{F}(U)$ such that $s\big|_{U_i} = s_i$ for all $i \in I$.

What Gluing means

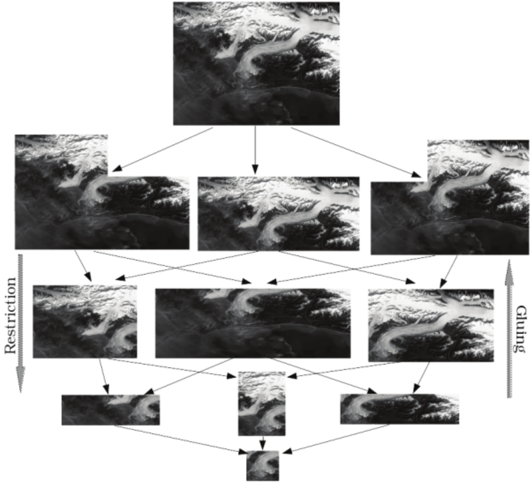


Figure: Example 127 – Sheaf Theory Through Examples

The Definition™

Not so scary is it!

Definition (Sheaf)

A *sheaf* on a topological space X is a presheaf (contravariant functor)

$$\mathcal{F} : \text{Open}(X)^{\text{op}} \to \text{Sets}$$

satisfying the following properties:

1. **Locality:** Suppose U is an open set, $\{U_i\}_{i \in I}$ is an open cover of U with $U_i \subseteq U$ for all $i \in I$, and $s, t \in \mathcal{F}(U)$ are sections. If $s\big|_{U_i} = t\big|_{U_i}$
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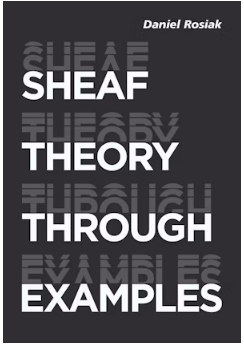
What are Sheaves for?

- Sheaves are a way to organise local and global information
- Category of Sheaves on a site (long story) form a Topos
- Sheaves are useful in Algebra for defining Cohomology theories (another long story)
- Everything in Prof. Coquand's talk

Figure: The Simpsons, S05E19

A Book Recommendation

- Book Website
- it's free



Now to hand you over to Quentin...